roots and the system is stable. Otherwise the system is unstable. The above steps exhaust all possibilities. A brief discussion of these is given below.

Discussion

Test step 1 is trivial. The truth of test step 2 follows from the decomposition Eq. (3) and the application of the well-known condition for asymptotic stability to $p(x)$ and $s(x^2)$. Step 3 is well-known. Step 5a is again an application of the necessary condition for asymptotic stability to $s(y)$. Step 5b is an application of a theorem of Fuller.[2] The method to determine $s(y)$ in step 4 and the recognition that its Hurwitz determinants are identical to Δ_{n-m} are new and recent results.[3,4] The proofs lie in the identification of the Hurwitz determinants with the resultants and subresultants of the two subpolynomials $h(x^2)$ and $g(x^2)$ of the characteristic polynomial $f(x)$. Although this identification was made independently by the author,[3] a literature search subsequently revealed Fuller[2] seems to be the first author to make this identification by going back to the almost forgotten early work of Trudi. Householder[4] also discussed this identification in connection with the more general problem of a complex characteristic polynomial using the theorems of Trudi and Netto. Although Householder's work is not addressed directly to the present problem, his result is applicable if one converts the real polynomial $f(x)$ to a complex polynomial $w(z)$ by the transformation $x = iz$. Finally it should be pointed out the test steps outlined are essentially based on the Routh-Hurwitz algorith. It is conceivable other equivalent algorithms such as those suggested by Duffin[5] may be used to advantage in some of the intermediate steps.

Example

$$f(x) = x^9 + 3x^8 + \tfrac{9}{2}x^7 + \tfrac{23}{2}x^6 + 7x^5 + 14x^4 + \tfrac{9}{2}x^3 + \tfrac{13}{2}x^2 + x + 1$$

For this example,

$$h(x^2) = 3x^8 + \tfrac{23}{2}x^6 + 14x^4 + \tfrac{13}{2}x^2 + 1$$

$$g(x^2) = x^8 + \tfrac{9}{2}x^6 + 7x^4 + \tfrac{9}{2}x^2 \pm 1$$

$$\Delta_{n-1} = \Delta_8 = \begin{vmatrix} 3 & 23/2 & 14 & 13/2 & 1 & 0 & 0 & 0 \\ 1 & 9/2 & 7 & 9/2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 23/2 & 14 & 13/2 & 1 & 0 & 0 \\ 0 & 1 & 9/2 & 7 & 9/2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 23/2 & 14 & 13/2 & 1 & 0 \\ 0 & 0 & 1 & 9/2 & 7 & 9/2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 23/2 & 14 & 13/2 & 1 \\ 0 & 0 & 0 & 1 & 9/2 & 7 & 9/2 & 1 \end{vmatrix}$$

Straightforward computations give

$$\Delta_8 = \Delta_6 = \Delta_4 = 0,$$

$$\Delta_1 = 3 > 0,\ \Delta_2 = \begin{vmatrix} 3 & 23/2 \\ 1 & 9/2 \end{vmatrix} = 2 > 0,\ \Delta_3 = 2 > 0,$$

$$s(y = x^2) = 1/2 \begin{vmatrix} 3 & h(y) \\ 1 & g(y) \end{vmatrix} = y^3 + \tfrac{7}{2}y^2 + \tfrac{7}{2}y + 1 = x^6 + \tfrac{7}{2}x^4 + \tfrac{7}{2}x^2 + 1$$

An application of test step 5 shows the necessary and sufficient criteria for the stability of a system with the characteristic equation $X^6 + b_1X^4 + b_2X^2 + b_3 = 0$ are $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $b_1^2 - 3b_2 > 0$ and $b_2^2(b_1^2 - 4b_2) + b_1b_3(18b_2 - 4b_1^2) - 27b_3^2 > 0$. Obviously $s(x^2)$ given above satisfies these criteria and the system Eq. (5) is stable, although not asymptotically stable. It is now straightforward to show that indeed $f(x) = (x^3 + 3x^2 + x + 1)\cdot(x^6 + \tfrac{7}{2}x^4 + \tfrac{7}{2}x^2 + 1)$ and the Hurwitz determinants of $p(x) = x^3 + 3x^2 + x + 1$ agree with Δ_1, Δ_2, and Δ_3 given above.

References

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Mean Curvature of a Deformed Spherical Surface

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When studying the equilibrium configuration of liquid drops under the action of surface tension, it is necessary to compute the mean curvature, $H(\theta,\phi)$, at a given point on the surface. The difference in pressure across the surface of the drop, ΔP, is given by

$$\Delta p = \sigma H = \alpha/2(1/R_1 + 1/R_2)$$

where R_1 and R_2 are the principal radii of curvature at a point on the surface. An expression for $(1/R_1 + 1/R_2)$ is derived in Landau and Lifschitz[1] for the case of a surface given in spherical coordinates as $\eta(\theta,\phi)$. This expression, however, is a perturbation expansion about a sphere, good only to first order. In order to obtain higher order expansions we derive an exact analytical expression for the mean curvature using techniques of Differential Geometry.[2] Although useful in engineering applications, this result appears not to have been previously published.

Let the surface be given by the function $\eta(\theta,\phi)$ where θ,ϕ are the usual polar angles and η the distance from the origin. Then in Euclidean 3-space, the surface is represented by the vector function:

$$\mathbf{X}(\theta,\phi) = (\eta \sin\theta \cos\phi, \eta \sin\theta \sin\phi, \eta \cos\theta)$$

At a point on the surface there is a tangent plane spanned by the two vectors

$$\partial\mathbf{X}/\partial\theta \equiv \mathbf{X}_1 = ([\eta \cos\theta + \eta_\theta \sin\theta] \cos\phi, [\eta \cos\theta + \eta_\theta \sin\theta] \sin\phi, [\eta_\theta \cos\theta - \eta \sin\theta])$$

and

$$\partial\mathbf{X}/\partial\phi \equiv \mathbf{X}_2 = ([\eta_\theta \cos\phi - \eta \sin\phi] \sin\theta, [\eta_\phi \sin\phi + \eta \cos\phi] \sin\theta, [\eta_\phi \cos\theta])$$

A unit normal vector exists with respect to this plane. Since by convention a sphere has positive curvature the unit normal will be taken as pointing inwards, towards the center of the sphere. Thus the unit normal is:

$$\mathbf{X}_3 = -(\mathbf{X}_1 \times \mathbf{X}_2)/|\mathbf{X}_1 \times \mathbf{X}_2|$$

Received April 4, 1971; revision received June 4, 1971.
Index Category: Hydrodynamics.

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It can be shown that $|\mathbf{X}_1 + \mathbf{X}_2|^2 = EG - F^2$, where

$$E \equiv \mathbf{X}_1 \cdot \mathbf{X}_1 = \eta^2 + \eta_\theta{}^2$$

$$F \equiv \mathbf{X}_1 \cdot \mathbf{X}_2 = \eta_\theta \eta_\phi$$

$$G \equiv \mathbf{X}_2 \cdot \mathbf{X}_2 = \eta^2 \sin^2\theta + \eta_\phi{}^2$$

The second partial derivatives of $\mathbf{X}$ are also needed to compute H. Let

$$\mathbf{X}_{11} \equiv \partial^2 \mathbf{X}/\partial\theta^2,\ \mathbf{X}_{12} \equiv \partial^2 \mathbf{X}/\partial\theta\partial\phi,\ \mathbf{X}_{22} \equiv \partial^2 \mathbf{X}/\partial\phi^2$$

and let

$$L \equiv \mathbf{X}_{11} \cdot \mathbf{X}_3,\ M \equiv \mathbf{X}_{12} \cdot \mathbf{X}_3,\ N \equiv \mathbf{X}_{22} \cdot \mathbf{X}_3$$

Then

$$H(\theta,\phi) = \frac{1}{2}\left[\frac{1}{R_1(\theta,\phi)} + \frac{1}{R_2(\theta,\phi)}\right] = \frac{E\cdot N - 2F\cdot M + G\cdot L}{2(E\cdot G - F^2)}$$

Explicitly we compute:

$$2H = \frac{\left(\{\eta^2 [2\eta \sin\theta - (\partial/\partial\theta)(\eta_\theta \sin\theta)] + \eta_\theta{}^2 [2\eta \sin\theta - (\partial/\partial\theta)(\eta \cos\theta)]\} \sin^2\theta - \{[\partial/\partial\phi(\eta_\phi[\eta^2 + \eta_\theta{}^2]) - 2_{\phi\phi}(\eta^2 + \eta_\theta{}^2)] \sin\theta - \eta_\phi{}^2 \partial^2/\partial\theta^2(\eta \sin\theta)\}\right)}{\eta[(\eta^2 + \eta_\theta{}^2)\sin^2\theta + \eta_\phi{}^2]^{3/2}}$$

For the case of a sphere, $\eta = R$, $\eta_\phi = 0$, $\eta_\theta = 0$, this reduces to $H = 1/R$, as expected. To recover the first order expression in Ref.2 , we represent η as a perturbation expansion, $\eta(\theta,\phi) = R + \zeta(\theta,\phi)\epsilon \times 0(\epsilon^2)$. Dropping the ϵ explicitly, as is conventional, the above expression is linearized to

$$\sin^2\theta[-R^2\zeta_{\theta\theta} \sin\theta - R^2\zeta_\theta \cos\theta + 2R^3 \sin\theta + 6R^2\zeta \sin\theta] - R^2\zeta_{\phi\phi} \sin\theta/(R + \zeta)(R^2 + 2R\zeta)^{3/2} \sin^3\theta$$

Expanding the denominator, this becomes

$$\left[2R^3 + 6R^2\zeta - R^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\zeta}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\zeta}{\partial\phi^2}\right\}\right]\cdot\left[\frac{1}{R^4} - \frac{4\zeta}{R^5}\right] = \frac{2}{R} - \frac{2\zeta}{R^2} - \frac{1}{R^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\zeta}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\zeta}{\partial\phi^2}\right\}$$

which is the expression in Landau & Lifschitz.

References

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